DERIVED CATEGORIES AND HOCHSCHILD COHOMOLOGY

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Abstract. The goal of this seminar is to study Hochschild cohomology for algebras towards applications mostly in Representation Theory and in Algebraic Geometry. It is well known that Hochschild cohomology of algebraic (rings, algebras etc) or geometric objects (varieties, schemes etc) records very important information about the particular objects and it is connected to various homological and geometrical properties including the theory of deformations and support varieties. Our aim in this working group is to study the basic theory of Hochschild cohomology for algebras and then move to applications in the theory of support varieties for finite dimensonal algebras. Moreover, we will also outline the basic theory of derived categories and prove (at the very end of the seminar) that the Hochschild cohomology is an invariant under derived equivalences. In this seminar we will mainly follow the book of Sarah J. Witherspoon [8] on Hochschild cohomology and below we state the contents of this working group.

Lecture 1 (04.03.2022): Introduction and quick overview

This talk will give a quick overview of Hochschild cohomology of algebras. Motivation for this cohomology theory will be given and we will explain why Hochschild cohomology is essential in various areas of representation theory and algebraic geometry. We will define Hochschild homology and cohomology, we will provide examples, and we will explain the interpretation in low degrees.
References: Sections 1.1, 1.2 and 1.3 from [8].
Speaker: Chrysostomos Psaroudakis

Lecture 2 (11.03.2022): Graded commutative algebras and Gerstenhaber algebras

In this talk, using the cup product, we will prove that the Hochschild cohomology is a graded commutative algebra. Moreover, we will show that the Hochschild cohomology of an algebra is a Gerstenhaber algebra. We will then explain that the cap product provides a pairing between Hochschild cohomology and Hochschild homology, the shuffle product will be introduced and then we will explain how the Hochschild homology of an algebra is a graded commutative algebra under the shuffle product. Time permitting, we will discuss the Hodge decomposition of Hochschild cohomology.
References: Sections 1.4, 1.5 and 1.6 from [8].
Speaker: Odysseas Giatagantzidis

Lectures 3 and 4 (18.03.2022 and 25.03.2022): Yoneda product and actions

This talk will first provide the definition of Yoneda product on Hochschild cohomology, and then we will prove that for the bar resolution the Yoneda product is equivalent to the cup product. We will discuss tensor product of complexes and then using the Yoneda composition we will show that the graded vector space $\mathsf{Ext}^*_A(M,N)$, where A is an algebra and M, N two left A-modules, is a graded module over the Hochschild cohomology ring $\mathsf{HH}^*(A)$ of A. We will define the graded center of a graded algebra and we will prove that there is a ring homomorphism $\mathsf{HH}^*(A) \to \mathsf{Ext}^*_A(M,M)$ whose image is contained in the graded center of $\mathsf{Ext}^*_A(M,M)$.
References: [8, Chapter 2].
Speaker: Chrysostomos Psaroudakis

Lecture 5 (01.04.2022): Some first computations

In this talk we will compute the Hochschild cohomology of the tensor product algebra of two algebras. We will spend some time on specific examples where we will analyse all necessary computations. Time permitting we will discuss the twisted tensor product of algebras.
References: Sections 3.1 and 3.2 from [8].
Speaker: Panagiotis Kostas

Date: February 21, 2022.

Lecture 6 (08.04.2022): The Hochschild-Kostant-Rosenberg Theorem

The aim of this talk is to prove the celebrated Theorem of Hochschild-Kostant-Rosenberg [3] that describes the Hochschild cohomology and homology rings of smooth commutative algebras. We will first define Koszul complexes that are used in the proof. We will analyse the proof of the HKR theorem and we will basically conclude that in the smooth commutative case the Hochschild cohomology and homology are generated as algebras by their one degree components.

References: [8, Section 3.3].

Speaker: Panagiotis Kostas

Lecture 7 (15.04.2022): Calabi-Yau algebras and Van den Bergh Duality

In this talk using Hochschild cohomology we will define (Hochschild) dimension, smoothness, differential forms, and square-zero extensions. For some smooth algebras, in particular Calabi-Yau algebras (that we will introduce), we will prove a duality due to Van den Bergh between Hochschild homology and cohomology. The latter result is the analog of Poincare duality of the homology and cohomology groups for manifolds. Time permitting we will introduce Connes differential on Hochschild cohomology and show how it induces the Batalin-Vilkovisky operation.

References: [8, Chapter 4].

Speaker: TBA

Note: (i) *After Easter holidays we will continue with algebraic deformations and the theory of support varieties for finite dimensional algebras. The precise schedule will be announced on time.*

(ii) *The above time table is indicative and can be changed regarding the will of the participants (for instance, provide more details in certain sections of* [8]*). In any case, we will update the current program with any change we make.*

References

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