BERGMAN PROJECTION INDUCED BY RADIAL WEIGHT ACTING ON $L^\infty(\mathbb{D})$

JOSÉ ÁNGEL PELÁEZ

ABSTRACT. The purpose of these notes is to serve as introduction to the study of the Bergman projection induced by a radial weight. It has been written in such a way that the access to the topic would be as easy as possible for a researcher who is not familiar with the topic. The main aim is to present a detailed proof of the following result: Let ω be a radial weight. Then, the orthogonal Bergman projection P_ω is bounded from $L^\infty(\mathbb{D})$ to the classical Bloch space $\mathcal{B}$ if and only if

$$\sup_{0\leqslant r<1} \frac{\int_r^1 \omega(s)\,ds}{\int_{\frac{1+r}{2}}^1 \omega(s)\,ds} < \infty.$$

We will also present an extension of this result which have been recently published in [13, Theorem 1].

Contents

1. Introduction

Let $\mathcal{H}(\mathbb{D})$ denote the space of all analytic functions in the unit disc $\mathbb{D} = \{z : |z| < 1\}$. For $f \in \mathcal{H}(\mathbb{D})$ and $0 < r < 1$, let

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p\,dt\right)^{1/p}, \quad 0 < p < \infty,$$

denote the L^p-means of $|f|$ on the circle of radius r centered at the origin, and let

$$M_\infty(r,f) = \sup_{|z|=r} |f(z)|$$

stand for the maximum modulus function. For $0 < p \leqslant \infty$, the Hardy space H^p consists of $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(r,f) < \infty.$$

Standard references for the theory of Hardy spaces are [2, 4].

A nonnegative integrable function ω on the unit disc $\mathbb{D}$ is called a weight. It is radial if $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$. For $0 < p < \infty$ and a weight ω, the weighted Lebesgue space L^p_ω consists of (Lebesgue) measurable functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|^p_{L^p_\omega} = \int_{\mathbb{D}} |f(z)|^p \omega(z)\,dA(z) < \infty,$$

Date: June 18, 2023.

where

$$dA(z) = \frac{1}{\pi} dx\, dy = d\theta r\, dr, \quad z == x + iy = re^{i\theta},$$

is the normalized element of the Lebesgue area measure on $\mathbb{D}$. The Bergman space A^p_ω is $L^p_\omega \cap \mathcal{H}(\mathbb{D})$.

Throughout this paper we assume $\widehat{\omega}(z) = \int_{|z|}^1 \omega(s)\, ds > 0$ for all $z \in \mathbb{D}$, for otherwise $A^p_\omega = \mathcal{H}(\mathbb{D})$. We shall use that $\widehat{\omega}(r)$ is a non-increasing function in $[0,1)$. As usual, we write L^p_α and A^p_α for the standard weighted Lebesgue and Bergman spaces induced by the radial weight $(1+\alpha)(1-|z|^2)^\alpha$, where $-1 < \alpha < \infty$ [3, 5, 15]. We write $dA_\alpha(z) = 1+\alpha)(1-|z|^2)^\alpha dA(z)$.

For each measurable set $E \subset \mathbb{D}$, we denote $\omega(E) = \int_E \omega(z)\, dA(z)$ for short. We recall that the Bloch space $\mathcal{B}$ [1] consists of $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2) + |f(0)| < \infty.$$

The following result is proved in [15, Chapter 5].

Theorem A. *For any $-1 < \alpha < \infty$ let us consider the Bergman projection*

$$P_\alpha(f)(z) = \int_{\mathbb{D}} \frac{f(\zeta)}{(1-z\bar{\zeta})^{2+\alpha}}\, dA_\alpha(\zeta), \quad f \in L^1_\alpha, \quad z \in \mathbb{D}.$$

Then, $P_\alpha : L^\infty \to \mathcal{B}$ is bounded and onto.

Question 1: What does mean this result?

Lemma 1. *Let ω be a radial weight and $0 < p < \infty$. Then, $\|\cdot\|_{A^p_\omega}$ convergence implies uniform convergence on compact subsets of $\mathbb{D}$.*

Proof. By the well-known inequality [15, Theorem 9.1]

$$|f(u)|^p \leqslant \frac{\|f\|^p_{H^p}}{(1-|u|^2)} \leqslant \frac{\|f\|^p_{H^p}}{(1-|u|)}, \quad u \in \mathbb{D},$$

applied to dilated function $f\left(\frac{1+|z|}{2}u\right)$, $z \in \mathbb{D}$. It follows

$$\left| f\left(\frac{1+|z|}{2}u\right)\right|^p \leqslant \frac{M_p^p\left(\frac{1+|z|}{2}, f\right)}{1-|u|}, \quad u \in \mathbb{D}.$$

So, taking $u = \frac{2z}{1+|z|}$,

$$|f(z)|^p \leqslant 2\frac{M_p^p\left(\frac{1+|z|}{2}, f\right)}{1-|z|} \leqslant 4\frac{\int_{\frac{1+|z|}{2}}^1 M_p^p(s,f)s\omega(s)\, ds}{(1-|z|)\widehat{\omega}\left(\frac{1+|z|}{2}\right)} \leqslant 4\frac{\|f\|^p_{A^p_\omega}}{(1-|z|)\widehat{\omega}\left(\frac{1+|z|}{2}\right)}.$$

Therefore, if $0 < r < 1$,

$$\sup_{|z|\leqslant r} |f(z)|^p \leqslant 4\frac{\|f\|^p_{A^p_\omega}}{(1-r)\widehat{\omega}\left(\frac{1+r}{2}\right)},$$

which implies that uniform convergence on compact subsets of $\mathbb{D}$. □

Let τ_0 the induced topology by the uniform convergence on compact subsets of $\mathbb{D}$.

Corollary 2. *Let ω be a radial weight and $0 < p < \infty$. Then,*

(i) *The point evaluation $L_z f = f(z)$, $z \in \mathbb{D}$, are bounded linear functionals and A^2_ω is a closed subspace of L^2_ω. Therefore, there exist unique $\{B^\omega_z\}_{z\in\mathbb{D}} \subset A^2_\omega$ such that*

$$f(z) = \langle f, B^\omega_z\rangle_{A^2_\omega} = \int_{\mathbb{D}} f(\zeta)\overline{B^\omega_z(\zeta)}\,\omega(\zeta)dA(\zeta), \quad z \in \mathbb{D}, \quad f \in A^2_\omega. \tag{1.1}$$

The family of $\{B^\omega_z\}_{z\in\mathbb{D}}$ are called the (Bergman) reproducing kernels of A^2_ω.

(ii) *There exists the orthogonal projection P_ω from L^2_ω to A^2_ω. Moreover,*

$$P_\omega(f)(z) = \int_{\mathbb{D}} f(\zeta)\overline{B^\omega_z(\zeta)}\,\omega(\zeta)dA(\zeta), \quad f \in L^2_\omega. \tag{1.2}$$

Proof. (i). Apply Lemma 1 to the compact $\{z\} \subset \mathbb{D}$ to deduce that point evaluation $L_z f = f(z)$, $z \in \mathbb{D}$, are bounded linear functionals. A^2_ω is a closed subspace of L^2_ω by Lemma 1 and the fact that $(\mathcal{H}(\mathbb{D}), \tau_0)$ is complete. Then Riesz representation theorem applied to the Hilbert space A^2_ω give the existence of (unique) $\{B^\omega_z\}_{z\in\mathbb{D}} \subset A^2_\omega$ such that (1.1) holds.

(ii). Since A^2_ω is a closed subspace of L^2_ω, there exists the orthogonal projection P_ω from L^2_ω to A^2_ω. Moreover, since $P_\omega = P^\star_\omega$

$$\begin{aligned} P_\omega(f)(z) &= \langle P_\omega(f), B^\omega_z\rangle_{L^2_\omega} \\ &= \langle f, P_\omega(B^\omega_z)\rangle_{L^2_\omega} \\ &= \langle f, B^\omega_z\rangle_{L^2_\omega} = \int_{\mathbb{D}} f(\zeta)\overline{B^\omega_z(\zeta)}\,\omega(\zeta)dA(\zeta), \quad z \in \mathbb{D}, \quad f \in L^2_\omega. \end{aligned}$$

□

Question 2:: Which are the radial weights such that $P_\omega : L^\infty \to \mathcal{B}$ is bounded and/or onto?

Theorem 3. *Let $(X, \|\cdot\|_X)$ a Hilbert space of analytic functions such that for any compact subset $K \subset \mathbb{D}$ there is a constant $C = C(K, X) > 0$ such that*

$$\sup_{z\in K} |f(z)| \leqslant C\|f\|_X, \quad f \in X.$$

Then, the reproducing kernels $\{K^X_a\}_{\alpha\in\mathbb{D}}$ satisfy

$$K^X_a(z) = \sum_{n=0}^{\infty} e_n(z)\overline{e_n(a)}, \quad a, z \in \mathbb{D},$$

for any orthonormal basis $\{e_n\}_{n=0}^\infty$ of X. Moreover, the convergence is absolute and uniform on compact subsets of $\mathbb{D}$.

Proof. Follow the lines of [15, Theorem 4.19]. □

Consequently, the Bergman reproducing kernels $\{B^\omega_z\}_{z\in\mathbb{D}}$, induced by a radial weight ω, can be written as $B^\omega_z(\zeta) = \sum \overline{e_n(z)}e_n(\zeta)$ for each orthonormal basis $\{e_n\}$ of A^2_ω, and therefore using the basis induced by the normalized monomials $e_n(z) = \frac{z^n}{\|z^n\|_{A^2_\omega}} = \frac{z^n}{\sqrt{2\int_0^1 r^{2n+1}\omega(r)\,dr}}$, $n \in \mathbb{N}\cup\{0\}$,

$$B^\omega_z(\zeta) = \sum_{n=0}^{\infty} \frac{(\overline{z}\zeta)^n}{2\omega_{2n+1}}, \quad z, \zeta \in \mathbb{D}. \tag{1.3}$$

Here ω_{2n+1} are the odd moments of ω, and in general from now on we write $\omega_x = \int_0^1 r^x\omega(r)\,dr$ for all $x \geqslant 0$. We shall use throughout these notes that

(i) $\omega_x \leqslant \omega_y$ for any $0 \leqslant y \leqslant x < \infty$;
(ii) For each $k > 0$ and ω a radial weight, there is $C = C(\omega, k)$ such that

$$\omega_{x+k} \leqslant \omega_x \leqslant C\omega_{x+k}, \quad x \geqslant 0.$$

If $\omega(z) = (\alpha + 1)(1 - |z|^2)^\alpha$, then (see [15, Corollary 4.20])

$$B^\omega_z(\zeta) = B^\alpha_z(\zeta) = \frac{1}{(1 - \bar{z}\zeta)^{2+\alpha}}, \quad z, \zeta \in \mathbb{D}.$$

By and large, the Bergman reproducing kernels B^ω_z do not have a neat formula, so we are forced to deal with the expression (1.3) to tackle our question.

Lemma B. Hardy's inequality. [2, p. 48]. *For any* $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n \in \mathcal{H}(\mathbb{D})$,

$$\sum_{n=0}^{\infty} \frac{|\widehat{f}(n)|}{n+1} \leqslant \pi \|f\|_{H^1}.$$

The next result is known for experts in the topic, however, as fas as I know, the next proof does not appear in any published paper.

Proposition 4. *Let* ω *be a radial weight, then* P_ω *is not bounded from* $L^\infty(\mathbb{D})$ *to* H^∞.

Proof. The operator P_ω is bounded from $L^\infty(\mathbb{D})$ to H^∞ if there is $C = C(\omega) > 0$ such that

$$\sup_{z\in\mathbb{D}} \left| \int_{\mathbb{D}} f(\zeta) \overline{B_z^\omega(\zeta)}\, \omega(\zeta) dA(\zeta) \right| \leqslant C \|f\|_{L^\infty(\mathbb{D})}, \quad f \in L^\infty(\mathbb{D}).$$

For each $a \in \mathbb{D}$, consider the function

$$f_a(\zeta) = \begin{cases} \frac{B_a^\omega(\zeta)}{|B_a^\omega(\zeta)|} & \text{if } B_a^\omega(\zeta) \neq 0 \\ 0 & \text{if } B_a^\omega(\zeta) = 0 \end{cases}.$$

Then, for each $a \in \mathbb{D}$

$$\begin{aligned} \int_{\mathbb{D}} |B_a^\omega(\zeta)|\, \omega(\zeta) dA(\zeta) &= \int_{\mathbb{D}} f_a(\zeta) \overline{B_a^\omega(\zeta)}\, \omega(\zeta) dA(\zeta) \\ &\leqslant \sup_{z\in\mathbb{D}} \left| \int_{\mathbb{D}} f_a(\zeta) \overline{B_z^\omega(\zeta)}\, \omega(\zeta) dA(\zeta) \right| \leqslant C \|f_a\|_{L^\infty(\mathbb{D})} = C. \end{aligned}$$

That is,

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |B_a^\omega(\zeta)|\, \omega(\zeta) dA(\zeta) < \infty. \tag{1.4}$$

On the other hand, it is clear that P_ω is bounded from $L^\infty(\mathbb{D})$ to H^∞ if (1.4) holds.

Let us see that (1.4) never happens. Using (1.3) and Hardy's inequality, for each $a \in \mathbb{D} \backslash \{0\}$

$$\begin{aligned} \int_{\mathbb{D}} |B_a^\omega(\zeta)|\, \omega(\zeta) dA(\zeta) &= 2 \int_0^1 M_1(B_a^\omega, s) s\omega(s)\, ds \\ &\geqslant 2\pi \int_0^1 \left(\sum_{n=0}^{\infty} \frac{s^n |a|^n}{2(n+1)\omega_{2n+1}} \right) s\omega(s)\, ds \\ &= \pi \sum_{n=0}^{\infty} \frac{|a|^n}{(n+1)\omega_{2n+1}} \int_0^1 s^{n+1} \omega(s)\, ds \\ &= \pi \sum_{n=0}^{\infty} \frac{|a|^n \omega_{n+1}}{(n+1)\omega_{2n+1}} \geqslant \pi \sum_{n=0}^{\infty} \frac{|a|^n}{n+1} = \frac{\pi}{|a|} \log\left(\frac{1}{1-|a|} \right), \end{aligned}$$

Therefore, $\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |B_a^\omega(\zeta)|\, \omega(\zeta) dA(\zeta) = \infty$. This finishes the proof. □

2. A necessary condition. The class of radial upper doubling weights

Throughout these notes, the letter $C = C(\cdot)$ will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. If there exists a constant $C = C(\cdot) > 0$ such that $a \leqslant Cb$, we will write either $a \lesssim b$ or $b \gtrsim a$. In particular, if $a \lesssim b$ and $a \gtrsim b$, then we will write $a \asymp b$.

Proposition 5. [13, Theorem 1] *Let* ω *be a radial weight such that* $P_\omega : L^\infty(\mathbb{D}) \to \mathcal{B}$ *is bounded, then*

$$\sup_{n\in\mathbb{N}} \frac{\omega_n}{\omega_{2n}} < \infty.$$

Proof. The boundednes of P_ω from $L^\infty(\mathbb{D})$ to H^∞ is equivalent to the inequality

$$\left|\int_{\mathbb{D}} f(\zeta)\overline{B_0^\omega(\zeta)}\,\omega(\zeta)dA(\zeta)\right|+\sup_{z\in\mathbb{D}}(1-|z|^2)\left|\int_{\mathbb{D}} f(\zeta)\left(\overline{B_z^\omega(\zeta)}\right)'\omega(\zeta)dA(\zeta)\right|\leqslant C(\omega)\|f\|_{L^\infty(\mathbb{D})},\quad f\in L^\infty(\mathbb{D}).$$

So, arguing as in Proposition 4, this is equivalent to

$$\|P_\omega\|_{L^\infty(\mathbb{D})\to\mathcal{B}}\asymp\|B_0^\omega\|_{A^1_\omega}+\sup_{z\in\mathbb{D}}(1-|z|^2)\int_{\mathbb{D}}\left|(B_\zeta^\omega)'(z)\right|\omega(\zeta)\,dA(\zeta). \tag{2.1}$$

Therefore, (2.1), (1.3) and Hardy's inequality yield

$$\begin{aligned}\infty&>\sup_{z\in\mathbb{D}}(1-|z|^2)\int_{\mathbb{D}}\left|(B_\zeta^\omega)'(z)\right|\omega(\zeta)\,dA(\zeta)\gtrsim\sup_{z\in\mathbb{D}}(1-|z|^2)\sum_{n=1}^\infty\frac{|z|^{n-1}n}{\omega_{2n+1}}\frac{1}{n}\int_0^1 r^{n+1}\omega(r)\,dr\\&\geqslant\sup_{z\in\mathbb{D}}(1-|z|)\left|\sum_{n=0}^\infty\frac{\omega_{n+2}}{\omega_{2n+3}}z^n\right|\geqslant\sup_{N\in\mathbb{N}}\frac{1}{N}\sum_{n=0}^N\frac{\omega_{n+2}}{\omega_{2n+3}}\left(1-\frac{1}{N}\right)^n\gtrsim\sup_{N\in\mathbb{N}}\frac{1}{N}\sum_{n=0}^N\frac{\omega_{n+2}}{\omega_{2n+3}}.\end{aligned}$$

For $x\in\mathbb{R}$, let $E[x]\in\mathbb{Z}$ be defined by $E[x]\leqslant x<E[x]+1$. Since

$$1\gtrsim\frac{1}{N}\sum_{n=0}^N\frac{\omega_{n+2}}{\omega_{2n+3}}\geqslant\frac{\omega_{N+2}}{N}\sum_{n=E[\frac{3N}{4}]}^N\frac{1}{\omega_{2n+3}}\geqslant\frac{\omega_{N+2}}{4\omega_{2E[\frac{3N}{4}]+3}},\quad N\in\mathbb{N},$$

there exists a constant $C=C(\omega)>0$ such that

$$\begin{aligned}\omega_{N+2}&\leqslant C\omega_{2E[\frac{3N}{4}]+3}\leqslant C^2\omega_{2E[\frac{3}{4}(2E[\frac{3N}{4}]+1)]+3}\\&\leqslant C^2\omega_{2E[\frac{9N}{8}-\frac{3}{4}]+3}\leqslant C^2\omega_{\frac{9N}{4}-\frac{1}{2}}\leqslant C^2\omega_{2N+4},\quad N\geqslant 18,\end{aligned}$$

that is, $\sup_{n\in\mathbb{N}}\frac{\omega_n}{\omega_{2n}}<\infty$.

□

Question: Is the reverse of Proposition 5 true?

Conditions for radial weights ω provided in terms of moments ω_n, $n\in\mathbb{N}$, are useful for a theoretical purposes, however they may not be handy because obtaining asymptotic estimates of ω_n, as $n\to\infty$, may be a tough question for a general radial weight. Therefore, in order to answer the above question, we will reformulate the condition $\sup_{n\in\mathbb{N}}\frac{\omega_n}{\omega_{2n}}<\infty$.

The class $\widehat{\mathcal{D}}$ consists of radial weights ω such that the tail integral

$$\widehat{\omega}(z)=\int_{|z|}^1\omega(s)\,ds,\quad z\in\mathbb{D},$$

is doubling, that is, there exists $C=C(\omega)\geqslant 1$ such that

$$\widehat{\omega}(r)\leqslant C\widehat{\omega}\left(\frac{1+r}{2}\right),\quad 0\leqslant r<1. \tag{2.2}$$

Lemma 6. [8, 14]. *Let ω be a radial weight such that $\widehat{\omega}(r)>0$ for all $0\leqslant r<1$. Then the following statements are equivalent:*

(i) $\omega\in\widehat{\mathcal{D}}$;

(ii) *There exist $C=C(\omega)\geqslant 1$ and $\beta=\beta(\omega)>0$ such that*

$$\widehat{\omega}(r)\leqslant C\left(\frac{1-r}{1-t}\right)^\beta\widehat{\omega}(t),\quad 0\leqslant r\leqslant t<1;$$

(iii) *There exist $C=C(\omega)>0$ and $\gamma=\gamma(\omega)>0$ such that*

$$\int_0^t\frac{\omega(s)}{(1-s)^\gamma}\,ds\leqslant C\frac{\widehat{\omega}(t)}{(1-t)^\gamma},\quad 0\leqslant t<1;$$

(iv) *There exists a constant $C=C(\omega)>0$ such that*

$$\int_0^t s^{\frac{1}{1-t}}\omega(s)\,ds\leqslant C\widehat{\omega}(t),\quad 0\leqslant t<1;$$

(v) *There exists a constant* $C = C(\omega) > 0$ *such that*

$$\omega_x \leqslant C\widehat{\omega}\left(1-\frac{1}{x}\right), \quad 1 \leqslant x < \infty;$$

(vi) *There exists* $\lambda = \lambda(\omega) \geqslant 0$ *such that*

$$\int_{\mathbb{D}} \frac{\omega(z)}{|1-\overline{\zeta}z|^{\lambda+1}}\, dA(z) \asymp \frac{\widehat{\omega}(\zeta)}{(1-|\zeta|)^{\lambda}}, \quad \zeta \in \mathbb{D};$$

(vii) *The weight*

$$\omega^{\star}(z) = \int_{|z|}^{1} \log\frac{s}{|z|}\omega(s)s\, ds, \quad z \in \mathbb{D}\backslash\{0\};$$

satisfies

$$\omega^{\star}(z) \asymp \widehat{\omega}(z)(1-|z|), \quad |z| \geqslant \frac{1}{2}.$$

(viii) *There exists* $C = C(\omega) > 0$ *such that* $\omega_n \leqslant C\omega_{2n}$ *for all* $n \in \mathbb{N}$*;*

(ix) *There exist* $C = C(\omega) > 0$ *and* $\eta = \eta(\omega) > 0$ *such that*

$$\omega_x \leqslant C\left(\frac{y}{x}\right)^{\eta}\omega_y, \quad 0 < x \leqslant y < \infty;$$

(x) *There exist* $K = K(\omega) > 1$ *and* $C = C(\omega, K) > 1$ *such that* $1-\rho_n(\omega,K) \geqslant C(1-\rho_{n+1}(\omega,K))$ *for all* $n \in \mathbb{N}\cup\{0\}$.

Observation: Look at the proof of the equivalence of the first five conditions and condition (viii).

Proof. We begin with showing that (i), (ii) and (iii) are equivalent. To to this, let first $\omega \in \widehat{\mathcal{D}}$. If $0 \leqslant r \leqslant t < 1$ and $r_n = 1-2^{-n}$ for all $n \in \mathbb{N}\cup\{0\}$, then there exist k and m such that $r_k \leqslant r < r_{k+1}$ and $r_m \leqslant t < r_{m+1}$. Therefore

$$\begin{aligned}\widehat{\omega}(r) &\leqslant \widehat{\omega}(r_k) \leqslant C\widehat{\omega}(r_{k+1}) \leqslant \cdots \leqslant C^{m-k+1}\widehat{\omega}(r_{m+1}) \leqslant C^{m-k+1}\widehat{\omega}(t)\\ &= C^2 2^{(m-k-1)\log_2 C}\widehat{\omega}(t) \leqslant C^2\left(\frac{1-r}{1-t}\right)^{\log_2 C}\widehat{\omega}(t), \quad 0 \leqslant r \leqslant t < 1,\end{aligned}$$

and hence (ii) is satisfied.

Assume next (ii). Let $x > 0$ and $\gamma = \beta(1+x)$, where $\beta = \beta(\omega) > 0$ is that of (ii). Then

$$\begin{aligned}\int_0^t \frac{\omega(s)}{(1-s)^{\gamma}}\, ds &= \int_0^t \frac{\widehat{\omega}(s)^{1+x}}{(1-s)^{\gamma}}\frac{\omega(s)}{\widehat{\omega}(s)^{1+x}}\, ds \lesssim \frac{\widehat{\omega}(t)^{1+x}}{(1-t)^{\gamma}}\int_0^t \frac{\omega(s)}{\widehat{\omega}(s)^{1+x}}\, ds\\ &\asymp \frac{\widehat{\omega}(t)^{1+x}}{(1-t)^{\gamma}}\frac{1}{\widehat{\omega}(t)^{x}} \asymp \frac{\widehat{\omega}(t)}{(1-t)^{\gamma}},\end{aligned}$$

and thus (iii) is satisfied for each $\gamma > \beta$.

Assume now that (iii) is satisfied. An integration by parts shows that

$$\int_0^t \frac{\omega(s)}{(1-s)^{\gamma}}\, ds = -\frac{\widehat{\omega}(t)}{(1-t)^{\gamma}} + \widehat{\omega}(0) + \gamma\int_0^t \frac{\widehat{\omega}(s)}{(1-s)^{\gamma+1}}\, ds,$$

and it follows that

$$\int_0^t \frac{\widehat{\omega}(s)}{(1-s)^{\gamma+1}}\, ds \lesssim \frac{\widehat{\omega}(t)}{(1-t)^{\gamma}}, \quad 0 \leqslant t < 1.$$

By applying this for $\frac{1+t}{2}$ in place of t, and by estimating the integral on the left downwards to the integral over $(0,t)$, we deduce

$$\frac{\widehat{\omega}\left(\frac{1+t}{2}\right)}{(1-t)^{\gamma}} \gtrsim \int_0^t \frac{\widehat{\omega}(s)}{(1-s)^{\gamma+1}}\, ds \geqslant \widehat{\omega}(t)\int_0^t \frac{ds}{(1-s)^{\gamma+1}} \asymp \frac{\widehat{\omega}(t)}{(1-t)^{\gamma}}, \quad 0 \leqslant t < 1.$$

It follows that $\omega \in \widehat{\mathcal{D}}$ by the definition. Thus (i), (ii) and (iii) are equivalent.

We next show that (iii) implies (iv). A simple calculation shows that

$$s^{x-1}(1-s)^\gamma \leqslant \left(\frac{x-1}{x-1+\gamma}\right)^{x-1}\left(\frac{\gamma}{x-1+\gamma}\right)^\gamma \leqslant \left(\frac{\gamma}{x-1+\gamma}\right)^\gamma$$

for all $0<s<1$ and $1<x<\infty$. Therefore (iii), with $t=1-\frac{1}{x}$, yields

$$\int_0^{1-\frac{1}{x}} s^x\omega(s)\,ds \leqslant \left(\frac{\gamma x}{x-1+\gamma}\right)^\gamma \int_0^{1-\frac{1}{x}} \frac{\omega(s)}{x^\gamma(1-s)^\gamma} s\,ds \lesssim \int_{1-\frac{1}{x}}^1 \omega(s)\,ds, \quad 1<x<\infty.$$

Thus (iv) is satisfied. Further, (iv) with $t=1-\frac{1}{x}$ implies

$$\omega_x = \left(\int_0^{1-\frac{1}{x}} + \int_{1-\frac{1}{x}}^1\right) s^x\omega(s)\,ds \asymp \int_0^{1-\frac{1}{x}} s^x\omega(s)\,ds + \widehat{\omega}\left(1-\frac{1}{x}\right) \lesssim \widehat{\omega}\left(1-\frac{1}{x}\right),$$

and hence (v) is satisfied. Furthermore, (v) with $x=\frac{2}{1-t}$ and the inequality $-\log t \leqslant \frac{1}{t}(1-t)$, valid for all $0<t\leqslant 1$, yields

$$\begin{aligned}\widehat{\omega}\left(\frac{1+t}{2}\right) &\gtrsim \int_0^1 s^{\frac{2}{1-t}}\omega(s)\,ds \geqslant \int_0^t s^{\frac{2}{1-t}}\omega(s)\,ds \geqslant t^{\frac{2}{1-t}}\widehat{\omega}(t) = e^{-\frac{2}{1-t}\log\frac{1}{t}}\widehat{\omega}(t)\\ &\geqslant e^{-\frac{2}{t}}\widehat{\omega}(t), \quad 0<t<1,\end{aligned}$$

and it follows that $\omega \in \widehat{\mathcal{D}}$ by the definition. Therefore (i)–(v) are equivalent.

We next observe that (iii) implies (vi). Namely, (iii) yields

$$\begin{aligned}\int_{\mathbb{D}} \frac{\omega(z)}{|1-\overline{\zeta}z|^{\gamma+1}}\,dA(z) &\asymp \int_0^1 \frac{s\omega(s)}{(1-|\zeta|s)^\gamma}\,ds = \left(\int_0^{|\zeta|} + \int_{|\zeta|}^1\right) \frac{s\omega(s)}{(1-|\zeta|s)^\gamma}\,ds\\ &\leqslant \int_0^{|\zeta|} \frac{\omega(s)}{(1-s)^\gamma}\,ds + \frac{\widehat{\omega}(\zeta)}{(1-|\zeta|)^\gamma} \lesssim \frac{\widehat{\omega}(\zeta)}{(1-|\zeta|)^\gamma}, \quad \zeta\in\mathbb{D},\end{aligned}$$

and thus (vi) is satisfied with $\lambda=\gamma$. Further, if (vi) is satisfied, then an integration by parts gives

$$\begin{aligned}\frac{\widehat{\omega}(\zeta)}{(1-|\zeta|)^\lambda} &\gtrsim \int_{\mathbb{D}} \frac{\omega(z)}{|1-\overline{\zeta}z|^{\lambda+1}}\,dA(z) \gtrsim \int_0^{|\zeta|} \omega(s)\frac{s}{(1-s|\zeta|)^\lambda}\,ds\\ &= -\frac{\widehat{\omega}(\zeta)|\zeta|}{(1-|\zeta|^2)^\lambda} + \int_0^{|\zeta|} \widehat{\omega}(s)\left(\frac{(1-s|\zeta|)^\lambda + \lambda(1-s|\zeta|)^{\lambda-1}|\zeta|}{(1-s|\zeta|)^{2\lambda}}\right)ds, \quad \zeta\in\mathbb{D},\end{aligned}$$

and it follows that

$$\frac{\widehat{\omega}(\zeta)}{(1-|\zeta|)^\lambda} \gtrsim |\zeta| \int_0^{|\zeta|} \frac{\widehat{\omega}(s)}{(1-|\zeta|s)^{\lambda+1}}\,ds.$$

By choosing $\zeta=\frac{1+t}{2}$ we deduce

$$\frac{\widehat{\omega}\left(\frac{1+t}{2}\right)}{(1-t)^\lambda} \gtrsim \int_0^{\frac{1+t}{2}} \frac{\widehat{\omega}(s)}{\left(1-\frac{1+t}{2}s\right)^{\lambda+1}}\,ds \geqslant \widehat{\omega}(t)\int_0^t \frac{ds}{\left(1-\frac{1+t}{2}s\right)^{\lambda+1}} \asymp \frac{\widehat{\omega}(t)}{(1-t)^\lambda}, \quad \frac{1}{2}\leqslant t<1,$$

and it follows that $\omega\in\widehat{\mathcal{D}}$. Thus (i)–(vi) are equivalent.

To see that (vii) is an equivalent condition, we first observe that the inequalities $1-t\leqslant -\log t \leqslant (1-t)/t$, $0<t\leqslant 1$, yield

$$\int_r^1 (s-r)\omega(s)\,ds \leqslant \int_r^1 \log\frac{s}{r}\omega(s)s\,ds = \omega^\star(r) \leqslant \frac{1}{r}\int_r^1 (s-r)\omega(s)\,ds, \quad 0<r<1,$$

and hence $\omega^\star(r) \asymp \int_r^1 (s-r)\omega(s)\,ds$ for all $r \geqslant \frac{1}{2}$. Thus $\omega^\star(r) \lesssim \widehat{\omega}(r)(1-r)$ for all $r \geqslant \frac{1}{2}$. If $\omega \in \widehat{\mathcal{D}}$, then

$$\omega^\star(r) \gtrsim \int_{\frac{1+r}{2}}^1 (s-r)\omega(s)\,ds \geqslant \left(\frac{1+r}{2} - r\right)\widehat{\omega}\left(\frac{1+r}{2}\right) \asymp \widehat{\omega}(r)(1-r),$$

and thus (i) implies (vii). Conversely, if (vii) is satisfied, then there exists a constant $C = C(\omega) > 1$ such that

$$\widehat{\omega}(r)(1-r) \leqslant C\int_r^1 (s-r)\omega(s)\,ds, \quad \frac{1}{2} \leqslant r < 1.$$

Let $0 < p < \frac{1}{C-1}$, that is, $Cp < p+1$, and set $r_p = \frac{p+r}{p+1}$. Then

$$r_p - r = \frac{p+r}{p+1} - r = \frac{p+r-rp-r}{p+1} = \frac{p(1-r)}{p+1}$$

and

$$\begin{aligned}\widehat{\omega}(r)(1-r) &\leqslant C\int_r^{r_p} (s-r)\omega(s)\,ds + C\int_{r_p}^1 (s-r)\omega(s)\,ds\\ &\leqslant C\widehat{\omega}(r)\left(r_p - r\right) + C(1-r)\widehat{\omega}(r_p)\\ &= C\widehat{\omega}(r)\frac{p}{p+1}(1-r) + C\widehat{\omega}(r_p)(1-r)\end{aligned}$$

It follows that

$$\widehat{\omega}(r) \leqslant \frac{C(p+1)}{1+p-Cp}\widehat{\omega}(r_p), \quad \frac{1}{2} \leqslant r < 1.$$

If $C < 2$ we may take $p = 1$ and deduce $\omega \in \widehat{\mathcal{D}}$. For otherwise, fix $p > 0$ sufficiently small and use the argument employed in the proof of (i)⇒(ii) together with $1-r_p = (1-r)/(1+p) \asymp 1-r$ to obtain

$$\widehat{\omega}(r) \lesssim \widehat{\omega}\left(\frac{1+r}{2}\right), \quad \frac{1}{2} \leqslant r < 1.$$

This yields $\omega \in \widehat{\mathcal{D}}$, and thus (i)–(vii) are equivalent.

It is clear that (i) and (v) together imply (viii). Conversely, assume that (viii) is satisfied. Let $A = \sup_{n\in\mathbb{N}} \left(1 - \frac{1}{n+1}\right)^n < 1$, and fix $k \in \mathbb{N}$ large enough such that $C^k A^{2^k} < 1$. Then

$$\begin{aligned}\omega_n \leqslant C\omega_{2n} \leqslant C^k\omega_{2^k n} &= C^k\left(\int_0^{1-\frac{1}{n+1}} + \int_{1-\frac{1}{n+1}}^1\right) r^{2^k n}\omega(r)\,dr\\ &\leqslant C^k A^{2^k}\omega_n + C^k\widehat{\omega}\left(1 - \frac{1}{n+1}\right), \quad n \in \mathbb{N},\end{aligned}$$

and hence

$$\omega_n \leqslant \frac{C^k}{1 - C^k A^{2^k}}\widehat{\omega}\left(1 - \frac{1}{n+1}\right), \quad n \in \mathbb{N}.$$

If $n \leqslant x < n+1$, then

$$\int_0^1 s^x\omega(s)\,ds \leqslant \omega_n \lesssim \widehat{\omega}\left(1 - \frac{1}{n+1}\right) \leqslant \widehat{\omega}\left(1 - \frac{1}{x}\right),$$

and hence (v) follows. Thus (i)–(viii) are equivalent.

Assume now (viii) and let $1 \leqslant x \leqslant y < \infty$. Then there exist $n, m \in \mathbb{N} \cup \{0\}$ such that $n \leqslant x \leqslant n+1$ and $2^m n \leqslant y \leqslant 2^{m+1}n$. Hence (viii) yields

$$\begin{aligned}\omega_x \leqslant \omega_n \leqslant C^{m+1}\omega_{2^{m+1}n} &\leqslant 2^{(m+1)\log_2 C}\omega_y\\ &\leqslant \left(\frac{2y}{n+1}\frac{n+1}{n}\right)^{\log_2 C}\omega_y \leqslant C^2\left(\frac{y}{x}\right)^{\log_2 C}\omega_y,\end{aligned}$$

and thus (ix) follows. The choice $y = 2n = 2x$ in (ix) immediately gives (viii), and thus (viii) and (ix) are equivalent.

It remains to show that (x) is equivalent to the other conditions. To see this, assume first that there exist $K = K(\omega) > 1$ and $C = C(\omega) > 1$ such that $1 - \rho_n \geqslant C(1 - \rho_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$. Let $0 \leqslant r \leqslant t < 1$ and fix $n, k \in \mathbb{N} \cup \{0\}$ such that $\rho_n \leqslant r < \rho_{n+1}$ and $\rho_k \leqslant t < \rho_{k+1}$. Then

$$\begin{aligned} 1 - r &\geqslant 1 - \rho_{n+1} \geqslant C(1 - \rho_{n+2}) \geqslant \cdots \geqslant C^{k-n-1}(1 - \rho_k) \\ &\geqslant C^{-2} \left(\frac{K^{-n}}{K^{-(k+1)}} \right)^{\log_K C} (1 - t) \geqslant C^{-2} \left(\frac{\widehat{\omega}(r)}{\widehat{\omega}(t)} \right)^{\log_K C} (1 - t), \end{aligned}$$

and hence

$$\widehat{\omega}(r) \leqslant C^{\frac{2}{\log_K C}} \left(\frac{1 - r}{1 - t} \right)^{\frac{1}{\log_K C}} \widehat{\omega}(t), \quad 0 \leqslant r \leqslant t < 1,$$

and thus (ii) is satisfied. Conversely, by choosing $t = \rho_{n+1}$ and $r = \rho_n$ in (ii), we deduce $1 - \rho_{n+1} \leqslant \left(\frac{C}{K}\right)^{\frac{1}{\beta}} (1 - \rho_n)$, and (x) follows by choosing $K > C$. The proof of the lemma is now complete. □

3. Main result

Our main aim is to prove the following.

Theorem 7. [11, Theorem 1] *Let ω be a radial weight. Then, $P_\omega : L^\infty(\mathbb{D}) \to \mathcal{B}$ is bounded if and only if $\omega \in \widehat{\mathcal{D}}$.*

Bearing in mind, Lemma 6, Proposition 5 and (2.1), it is enough to prove

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) \int_{\mathbb{D}} \left| (B^\omega_\zeta)'(z) \right| \omega(\zeta) \, dA(\zeta) < \infty$$

holds if $\omega \in \widehat{\mathcal{D}}$. By using $z(B^\omega_\zeta)'(z) = \overline{\zeta (B^\omega_z)'(\zeta)}$, this is equivalent to prove that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) \int_{\mathbb{D}} \left| (B^\omega_\zeta)'(z) \right| \omega(\zeta) \, dA(\zeta) < \infty. \tag{3.1}$$

Consequently, Theorem 7 will be obtained joining Lemma 6, Proposition 5 and the next result.

Proposition 8. *Let $\omega \in \widehat{\mathcal{D}}$. Then,* (3.1) *holds.*

A proof of Proposition 8 follows from [11, Theorem 1]. In these notes, we will present an unpublished direct proof, which uses the same circles of ideas. In order to get this aim we introduce some notation and results.

The Riesz projection (that is the orthogonal projection $R : L^2(\partial\mathbb{D}) \to H^2$) given by

$$Rf(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(e^{it}}{1 - e^{-it} z} \, dt, \quad f \in L^1(\partial\mathbb{D}), \quad z \in \mathbb{D},$$

is a bounded operator from $L^p(\partial\mathbb{D})$ to H^p if and only if $1 < p < \infty$ [4, Chapter 3].

As a by product of this result we can obtain the following.

Corollary 9. *Let $1 < p < \infty$, then there is a constant $C(p) > 0$ such that*

$$\left\| \sum_{k \in J}^{\infty} \widehat{f}(k) z^k \right\|_{H^p} \leqslant C(p) \|f\|_{H^p},$$

for any $f(z) = \sum_{k=0}^\infty \widehat{f}(k) z^k \in \mathcal{H}(\mathbb{D})$ and any subset J of $\mathbb{N} \cup \{0\}$.

Applications of Corollary 9 for $J = I_n = \{k \in \mathbb{N} \cup \{0\} : 2^n \leqslant k < 2^{n+1}\}$, $\mathbb{N} \cup \{0\}$, are really useful for the study of function theory and operator theory on spaces of analytic functions in $\mathbb{D}$. Corollary 9 does not remain true for $0 < p \leqslant 1$. With the aim of presenting a "substitute" of Corollary 9 for $0 < p \leqslant 1$, we introduce the following notation and results.

3.1. **Estimates of the integral means the derivative of the Bergman reproducing kernel.** Let $W(z) = \sum_{k\in J} b_k z^k$ be a polynomial, where J denote a finite subset of $\mathbb{N}\cup\{0\}$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{H}(\mathbb{D})$. The Hadamard product

$$(W * f)(z) = \sum_{k=0}^{\infty} a_k b_k z^k, \quad z \in \mathbb{D},$$

is well defined. Furthermore, it is easy to observe that

$$(W * f)(e^{it}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} W(e^{i(t-\theta)}) f(e^{i\theta})\, d\theta. \tag{3.2}$$

For a given C^∞-function $\Phi : \mathbb{R} \to \mathbb{C}$ with compact support, set

$$A_{\Phi,m} = \max_{x\in\mathbb{R}} |\Phi(x)| + m \max_{x\in\mathbb{R}} |\Phi^{(m)}(x)|, \quad m \in \mathbb{N}\cup\{0\},$$

and define the polynomials

$$W_n^\Phi(z) = \sum_{k\in\mathbb{Z}} \Phi\left(\frac{k}{n}\right) z^k, \quad n \in \mathbb{N}. \tag{3.3}$$

The next result can be found in [7, pp. 111–113].

Theorem C. *Let $\Phi : \mathbb{R} \to \mathbb{C}$ be a compactly supported C^∞-function. Then the following statements hold:*

(i) *There exists a constant $C > 0$ such that*

$$|W_n^\Phi(e^{i\theta})| \leqslant C \min\left\{ n \max_{s\in\mathbb{R}} |\Phi(s)|, n^{1-m}|\theta|^{-m} \max_{s\in\mathbb{R}} |\Phi^{(m)}(s)| \right\}$$

for all $m \in \mathbb{N}\cup\{0\}$, $n \in \mathbb{N}$ and $0 < |\theta| < \pi$.

(ii) *If $0 < p \leqslant 1$ and $m \in \mathbb{N}$ with $mp > 1$, there exists a constant $C = C(p) > 0$ such that*

$$\left(\sup_n |(W_n^\Phi * f)(e^{i\theta})|\right)^p \leqslant C A_{\Phi,m}^p M(|f|^p)(e^{i\theta})$$

for all $f \in H^p$, where M denotes the Hardy-Littlewood maximal-operator

$$M(f)(e^{i\theta}) = \sup_{0<h<\pi} \frac{1}{2h}\int_{\theta-h}^{\theta+h} |f(e^{it})| dt.$$

(iii) *For each $0 < p < \infty$ and $m \in \mathbb{N}$ with $mp > 1$, there exists a constant $C = C(p) > 0$ such that*

$$\|W_n^\Phi * f\|_{H^p} \leqslant C A_{\Phi,m} \|f\|_{H^p}$$

for all $f \in H^p$ and $n \in \mathbb{N}$.

A particular case of the previous construction is useful for our purposes. By following [6, Section 2], let $\Psi : \mathbb{R} \to \mathbb{R}$ be a C^∞-function such that $\Psi \equiv 1$ on $(-\infty, 1]$, $\Psi \equiv 0$ on $[2, \infty)$ and Ψ is decreasing and positive on $(1, 2)$. Set $\psi(t) = \Psi\left(\frac{t}{2}\right) - \Psi(t)$ for all $t \in \mathbb{R}$. Let $V_0(z) = 1 + z$ and

$$V_n(z) = W_{2^{n-1}}^\psi(z) = \sum_{k=0}^{\infty} \psi\left(\frac{k}{2^{n-1}}\right) z^k = \sum_{k=2^{n-1}}^{2^{n+1}-1} \psi\left(\frac{k}{2^{n-1}}\right) z^k, \quad n \in \mathbb{N}. \tag{3.4}$$

These polynomials have the following properties (see [6, p. 175–177]):

$$\begin{aligned} & f(z) = \sum_{n=0}^{\infty} (V_n * f)(z), \quad f \in \mathcal{H}(\mathbb{D}), \\ & \|V_n * f\|_{H^p} \leqslant C\|f\|_{H^p}, \quad f \in H^p, \quad 0 < p < \infty, \\ & \|V_n\|_{H^p} \asymp 2^{n(1-1/p)}, \quad 0 < p < \infty. \end{aligned} \tag{3.5}$$

Proposition 10. *Let $\omega \in \widehat{\mathcal{D}}$. Then*

$$M_1(r,(B_a^\omega)') \lesssim 1+\int_0^{r|a|} \frac{1}{\widehat{\omega}(t)(1-t)^2}\,dt, \quad a\in\mathbb{D},\, r\in[0,1).$$

Proof. Let us recall that

$$(B_a^\omega)'(z) = \bar{a}\sum_{n=0}^{\infty}\frac{(n+1)(\bar{a}z)^n}{2\omega_{2n+3}}, a, z\in\mathbb{D}.$$

Firtsly, assume $\frac{1}{2}\leqslant |a|, r<1$. Bearing in mind (3.5),

$$M_1(r,(B_a^\omega)')) \leqslant \sum_{n=0}^{\infty}\|V_n * ((B_a^\omega)'))_r\|_{H^1}, \tag{3.6}$$

On the one hand,

$$\begin{aligned}\|V_0 * \left((B_a^\omega)'\right)_r\|_{H^1} &\leqslant \left\|\frac{1}{2\omega_3}+\frac{3\bar{a}rz}{2\omega_5}\right\|_{H^1} \leqslant \frac{1}{2\omega_3}+\frac{3}{2\omega_5}\\ &\lesssim \frac{1}{\widehat{\omega}(r|a|)(1-r|a|)}\int_{\frac{4r|a|-1}{3}}^{r|a|} dt\\ &\leqslant \frac{1}{\widehat{\omega}(r|a|)(1-r|a|)^2}\int_{\frac{4r|a|-1}{3}}^{r|a|} dt\\ &\asymp \int_{\frac{4r|a|-1}{3}}^{r|a|}\frac{1}{\widehat{\omega}(t)(1-t)^2}dt \leqslant \int_0^{r|a|}\frac{1}{\widehat{\omega}(t)(1-t)^2}\,dt \quad r,|a|\geqslant \frac{1}{2}.\end{aligned} \tag{3.7}$$

Analogously,

$$\|V_1 * \left((B_a^\omega)'\right)_r\|_{H^1} \lesssim \int_0^{r|a|}\frac{1}{\widehat{\omega}(t)(1-t)^2}dt, \quad r,|a|\geqslant\frac{1}{2}. \tag{3.8}$$

Now, for each $n\in\mathbb{N}$, $n\geqslant 2$, let us consider the following family of functions

$$\varphi_n(x) = \frac{(x+1)s^x}{\omega_{2x+3}}\chi_{[2^{n-1},2^{n+1}-1]}(x), \quad \frac{1}{4}\leqslant s<1,$$

for $x\in[2^{n-1},2^{n+1}-1)$. Observe that $\frac{d^k}{dx}(\omega_{2x+3}) = (\omega_{2x+3})^{(k)} = \int_0^1 s^{2x+3}(\log s)^k\omega(s)\,ds$, $k\in\mathbb{N}$. So there is an absolute $C>0$ such that

$$(\omega_{2x+3})^{(k)} \leqslant C\omega_{2x+2}, \quad x\geqslant 1, \quad k=1,2.$$

Bearing in mind this observation, a calculation shows that there is an absolute $C>0$ such that

$$|(\varphi_n)''(x)| \leqslant C\frac{(x+1)s^x}{\omega_{2x+3}}, \quad x\in[2^{n-1},2^{n+1}-1).$$

Therefore, we may choose C^∞-functions $\Phi_{1,n}$ with compact support contained in $[2^{n-2},2^{n+2}]$ such that $\Phi_n=\varphi_n$ in $[2^{n-1},2^{n+1}-1]$ and $C_1=C_1(\omega)>0$

$$\begin{aligned}A_{\Phi_n,2} &= \max_{x\in\mathbb{R}}|\Phi_n(x)| + 2\max_{x\in\mathbb{R}}|\Phi_n''(x)|\\ &\leqslant C_1\max_{x\in[2^{n-1},2^{n+1}-1)}\frac{(x+1)s^x}{\omega_{2x+3}}\\ &\leqslant C_1\frac{2^n s^{2^{n-1}}}{\omega_{2^n}}, \quad \frac{1}{2}\leqslant r,|a|<1.\end{aligned} \tag{3.9}$$

Then, if $a=|a|e^{i\theta}$ and $s=|a|r$,

$$V_n * \left((B_a^\omega)'\right)_r(z) = |a| \sum_{k=2^{n-1}}^{2^{n+1}-1} \psi\left(\frac{k}{2^{n-1}}\right) \frac{(k+1)(|a|r)^k}{\omega_{2k+3}} (ze^{-i\theta})^k$$
$$= |a| \sum_{k=2^{n-1}}^{2^{n+1}-1} \psi\left(\frac{k}{2^{n-1}}\right) \Phi_n(k)(ze^{-i\theta})^k.$$

Threfore, Theorem C(iii), (3.5) and (3.9) imply that there is $C=C(\mu,\omega)>0$ such that for each $n\in\mathbb{N}\backslash\{1\}$

$$\begin{aligned}\|V_n * \left((B_a^\omega)'\right)_r\|_{H^1} &\leqslant CA_{\Phi_n,2}\|V_n\|_{H^1}\\ &\leqslant CA_{\Phi_n,2}\\ &\leqslant C\frac{2^n s^{2^{n-1}}}{\omega_{2^n}}\\ &= C\frac{2^n(r|a|)^{2^{n-1}}}{\omega_{2^n}}, \quad \frac{1}{2}\leqslant r,|a|<1.\end{aligned}$$

Then by Lemma 11 (below, with $k=2$)

$$\begin{aligned}\sum_{n=2}^\infty \|V_n * (D^\mu(B_a^\omega))_r\|_{H^1} &\lesssim \sum_{n=2}^\infty \frac{2^n(r|a|)^{2^{n-1}}}{\omega_{2^n}}\\ &\lesssim \int_0^{r|a|} \frac{1}{\widehat{\omega}(t)(1-t)^2}dt, \quad r,|a|\geqslant\frac{1}{2}.\end{aligned} \tag{3.10}$$

By joining (3.6), (3.7), (3.8) and (3.10) we deduce

$$M_1(r,(B_a^\omega)')) \lesssim \int_0^{r|a|} \frac{1}{\widehat{\omega}(t)(1-t)^2}dt, \quad r,|a|\geqslant\frac{1}{2}. \tag{3.11}$$

Moreover, if $|a|\leqslant\frac{1}{2}$ or $r\leqslant\frac{1}{2}$,

$$M_1(r,(B_a^\omega)')) \leqslant \sum_{n=0}^\infty \frac{(n+1)}{2^{n+1}\omega_{2n+3}} \lesssim 1. \tag{3.12}$$

Thus (3.11) and (3.12) yields

$$M_1(r,(B_a^\omega)') \lesssim 1 + \int_0^{r|a|} \frac{1}{\widehat{\omega}(t)(1-t)^2}\,dt, \quad a\in\mathbb{D},\ r\in[0,1).$$

This finishes the proof. □

Proof of Proposition 8. For any $z\in\mathbb{D}\backslash\{0\}$, by Proposition 10

$$\begin{aligned}\int_{\mathbb{D}} \left|(B_\zeta^\omega)'(z)\right|\omega(\zeta)\,dA(\zeta) &\asymp \int_0^1 M_1(r,(B_z^\omega)')\omega(r)\,dr\\ &\lesssim 1+\int_0^1\left(\int_0^{r|z|}\frac{1}{\widehat{\omega}(t)(1-t)^2}\,dt\right)\omega(r)\,dr\\ &= 1+\int_0^{|z|}\frac{1}{\widehat{\omega}(t)(1-t)^2}\left(\int_{\frac{t}{|z|}}^1\omega(s)\,ds\right)\,dt\\ &\leqslant 1+\int_0^{|z|}\frac{dt}{(1-t)^2}\asymp\frac{1}{1-|z|}.\end{aligned}$$

This finishes the proof. □

In order to finish the proof of Proposition 10 we present the next technical result.

Lemma 11. [9, Lemma 6] *Let $\mu \in \widehat{\mathcal{D}}$, $\gamma > 0$, $m \geqslant 0$ and $k \in \mathbb{N}\backslash\{1\}$. Then,*

$$\sum_{n=0}^{\infty} \frac{k^{mn} r^{k^n}}{\mu_{k^n}^{\gamma}} \asymp \int_0^r \frac{dt}{(1-t)^{1+m}\widehat{\mu}(t)^{\gamma}}, \quad 0 \leqslant r < 1. \tag{3.13}$$

Proof. Since $\mu \in \widehat{\mathcal{D}}$,

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{k^{mn} r^{k^n}}{\mu_{k^n}^{\gamma}} &\asymp \frac{r}{\mu_1^{\gamma}} + \sum_{n=1}^{\infty} \frac{k^{mn} r^{k^n}}{\mu_{k^n}^{\gamma}} \frac{1}{k^n} \sum_{j=k^{n-1}}^{k^n-1} 1 \\ &\lesssim \frac{r}{\mu_1^{\gamma}} + \sum_{n=1}^{\infty} \sum_{j=k^{n-1}}^{k^n-1} \frac{r^j}{(j+1)^{1-m}\mu_j^{\gamma}} \\ &\asymp \sum_{j=1}^{\infty} \frac{r^j}{(j+1)^{1-m}\mu_{2j+1}^{\gamma}}, \quad 0 \leqslant r < 1.\end{aligned}$$

Analogously, it can be proved that

$$\sum_{n=0}^{\infty} \frac{k^{mn} r^{k^n}}{\mu_{k^n}^{\gamma}} \gtrsim \sum_{j=1}^{\infty} \frac{r^j}{(j+1)^{1-m}\mu_{2j+1}^{\gamma}}, \quad 0 < r < 1.$$

Now, let us prove that

$$\sum_{j=1}^{\infty} \frac{r^j}{(j+1)^{1-m}\mu_{2j+1}^{\gamma}} \asymp \int_0^r \frac{dt}{(1-t)^{m+1}\widehat{\mu}(t)^{\gamma}}, \quad 0 < r < 1, \tag{3.14}$$

and we will get (3.13).

Assume without loss of generality that $r \geqslant \frac{1}{2}$. Choose $N \in \mathbb{N}$ such that $1-\frac{1}{N} \leqslant r < 1-\frac{1}{N+1}$. Then, Lemma 6(v)-(ii) yields

$$\begin{aligned}\sum_{j=1}^{N} \frac{r^j}{(j+1)^{1-m}\mu_{2j+1}^{\gamma}} &\asymp \sum_{j=1}^{N} \frac{1}{(j+1)^{1-m}\widehat{\mu}\left(1-\frac{1}{2j+1}\right)^{\gamma}} \gtrsim \sum_{j=1}^{N} \int_j^{j+1} \frac{1}{s^{1-m}\widehat{\mu}\left(1-\frac{1}{s}\right)^{\gamma}}\, ds \\ &= \int_1^{N+1} \frac{1}{s^{1-m}\widehat{\mu}\left(1-\frac{1}{s}\right)^{\gamma}}\, ds = \int_0^{1-\frac{1}{N+1}} \frac{dt}{(1-t)^{m+1}\widehat{\mu}(t)^{\gamma}} \\ &\geqslant \int_0^r \frac{dt}{(1-t)^{m+1}\widehat{\mu}(t)^{\gamma}}.\end{aligned}$$

Since Lemma 6(v)-(ii) allows us to establish the same upper bound in a similar manner, we deduce

$$\sum_{j=1}^{N} \frac{r^j}{(j+1)^{1-m}\mu_{2j+1}^{\gamma}} \asymp \int_0^r \frac{dt}{(1-t)^{m+1}\widehat{\mu}(t)^{\gamma}}. \tag{3.15}$$

Lemma 6(ii) also implies

$$\frac{1}{(1-r)^m\widehat{\mu}(r)^{\gamma}} \asymp \int_{\frac{4r-1}{3}}^{r} \frac{dt}{(1-t)^{m+1}\widehat{\mu}(t)^{\gamma}}, \quad \frac{1}{2} \leqslant r < 1.$$

and the existence of a constant $M = M(p,\omega) > m$ such that $\frac{\widehat{\mu}(r)^\gamma}{(1-r)^M}$ is essentially increasing (take $M \geqslant \max\{\gamma\beta, m\}$). Hence,

$$\begin{aligned}\sum_{j=N+1}^{\infty} \frac{r^j}{(j+1)^{1-m}\mu_{2j+1}^\gamma} &\asymp \sum_{j=N+1}^{\infty} \frac{r^j}{(j+1)^{1-m}\widehat{\mu}\left(1-\frac{1}{j+1}\right)^\gamma} \\ &\lesssim \frac{1}{(N+1)^M\widehat{\mu}\left(1-\frac{1}{N+1}\right)^\gamma} \sum_{j=N+1}^{\infty} (j+1)^{M-m-1} r^j \\ &\asymp \frac{1}{(N+1)^m\widehat{\mu}\left(1-\frac{1}{N+1}\right)^\gamma} \\ &\asymp \frac{1}{(1-r)^m\widehat{\mu}(r)^\gamma} \\ &\asymp \int_{\frac{4r-1}{3}}^{r} \frac{dt}{(1-t)^{m+1}\widehat{\mu}(t)^\gamma} \\ &\lesssim \int_0^r \frac{dt}{(1-t)^{m+1}\widehat{\mu}(t)^\gamma}.\end{aligned}$$

and hence, this inequality together with (3.15) finishes the proof.

□

4. An extension of our main result

The question of which properties of a radial weight ω are determinative for $P_\omega : L^\infty(\mathbb{D}) \to \mathcal{B}$ to be bounded maybe paraphrased in terms of the dual space of A^1_ω because the dual $(A^1_\omega)^\star$ can be identified with $P_\omega(L^\infty(\mathbb{D}))$ under the A^2_ω-pairing

$$\langle f, g\rangle_{A^2_\omega} = \lim_{r\to 1^-} \int_{\mathbb{D}} f_r(z)\overline{g(z)}\omega(z)\,dA(z).$$

Our next result makes precise this statement.

Theorem 12. [11, Theorem 1] *Let ω be a radial weight. Then the following statements are equivalent:*

(i) $P_\omega : L^\infty(\mathbb{D}) \to \mathcal{B}$ *is bounded;*

(ii) *There exists $C = C(\omega) > 0$ such that for each $L \in (A^1_\omega)^\star$ there is a unique $g \in \mathcal{B}$ such that $L(f) = \langle f, g\rangle_{A^2_\omega}$ for all $f \in A^1_\omega$ with $\|g\|_{\mathcal{B}} \leqslant C\|L\|$, that is, $(A^1_\omega)^\star \subset \mathcal{B}$ via the A^2_ω-pairing;*

(iii) *There exists $C = C(\omega) > 0$ such that for each $L \in \mathcal{B}_0^\star$ there is a unique $g \in A^1_\omega$ such that $L(f) = \langle f, g\rangle_{A^2_\omega}$ for all $f \in \mathcal{B}_0$ with $\|g\|_{A^1_\omega} \leqslant C\|L\|$, that is, $\mathcal{B}_0^\star \subset A^1_\omega$ via the A^2_ω-pairing;*

(iv) $\omega \in \widehat{\mathcal{D}}$.

Before proving Theorem 12, let us recall the reader that $(A^1_\alpha)^\star \simeq \mathcal{B}$ and $\mathcal{B}_0^\star \simeq A^1_\alpha$ under the A^2_α-pairing [15, Chapter 5]. Here $\mathcal{B}_0$ stands for the little Bloch space which consists of $f \in \mathcal{H}(\mathbb{D})$ such that $f'(z)(1-|z|^2) \to 0$, as $|z| \to 1^-$.

Proof of Theorem 12. Theorem 7 gives (i)⇐(iv).

Next observe that for each radial weight ω the dual space $(A^1_\omega)^\star$ can be identified with $P_\omega(L^\infty(\mathbb{D}))$ under the A^2_ω-pairing, and in particular, for each $L \in (A^1_\omega)^\star$ there exists $h \in L^\infty(\mathbb{D})$ such that

$$L(f) = L_{P_\omega(h)}(f) = \langle f, P_\omega(h)\rangle_{A^2_\omega}, \quad f \in A^1_\omega, \quad \|L\| = \|h\|_{L^\infty(\mathbb{D})}. \tag{4.1}$$

To see this, note that

$$\langle f_r, P_\omega(h)\rangle_{L^2_\omega} = \langle f_r, h\rangle_{L^2_\omega}, \quad h \in L^1_\omega, \quad f \in \mathcal{H}(\mathbb{D}), \quad 0 < r < 1. \tag{4.2}$$

If now $h \in L^\infty(\mathbb{D})$ and $g = P_\omega(h)$, then

$$|L_g(f)| = \lim_{r\to 1^-} \left|\langle f_r, P_\omega(h)\rangle_{L^2_\omega}\right| = \lim_{r\to 1^-} \left|\langle f_r, h\rangle_{L^2_\omega}\right| \leqslant \|h\|_{L^\infty(\mathbb{D})}\|f\|_{A^1_\omega}, \quad f \in A^1_\omega, \tag{4.3}$$

and hence $L_g \in (A^1_\omega)^\star$ with $\|L_g\| \leqslant \|h\|_{L^\infty(\mathbb{D})}$. Conversely, if $L \in (A^1_\omega)^\star$, then by the Hahn-Banach theorem L can be extended to a bounded linear functional on L^1_ω with the same norm. Since $(L^1_\omega)^\star$ can be identified with $L^\infty(\mathbb{D})$ under the L^2_ω-pairing, there exists $h \in L^\infty(\mathbb{D})$ such that $L(f) = \langle f, h\rangle_{L^2_\omega}$, with $\|L\| = \|h\|_{L^\infty(\mathbb{D})}$, for all $f \in L^1_\omega$. In particular, for each $f \in A^1_\omega$ we have $L(f_r) = \langle f_r, h\rangle_{L^2_\omega} = \langle f_r, P_\omega(h)\rangle_{A^2_\omega}$ by (4.2). Therefore

$$L(f) = \lim_{r\to 1^-} L(f_r) = \lim_{r\to 1^-} \langle f_r, P_\omega(h)\rangle_{A^2_\omega} = L_{P_\omega(h)}(f), \quad f \in A^1_\omega. \tag{4.4}$$

Consequently if (i) holds, $L(f) = \langle f, h\rangle_{L^2_\omega}$, with $\|L\| = \|h\|_{L^\infty(\mathbb{D})}$. Then if $g = P_\omega(h)$,

$$\|g\|_{\mathcal{B}} = \|P_\omega(h)\|_{\mathcal{B}} \leqslant \|P_\omega\|\|h\|_{L^\infty(\mathbb{D})} = \|P_\omega\|\|L\|,$$

Conversely, let $h \in L^\infty(\mathbb{D})$, and consider the functional defined by $L_{P_\omega(h)}(f) = \langle f, P_\omega(h)\rangle_{A^2_\omega}$ for all $f \in A^1_\omega$. Then $L_{P_\omega(h)} \in (A^1_\omega)^\star$ with $\|L\| \leqslant \|h\|_{L^\infty(\mathbb{D})}$ by what we just proved. But by the hypothesis (ii), there exists $g \in \mathcal{B}$ such that $L_{P_\omega(h)}(f) = L_g(f)$ for all $f \in A^1_\omega$, and $\|g\|_{\mathcal{B}} \leqslant C\|L\|$. Hence $L_{P_\omega(h)-g}$ represents a zero functional, and thus $g = P_\omega(h)$ with $\|P_\omega(h)\|_{\mathcal{B}} = \|g\|_{\mathcal{B}} \leqslant C\|L\| = C\|h\|_{L^\infty(\mathbb{D})}$. Therefore (i) is satisfied.

We now show that (iii) and (iv) are equivalent. Assume first $\omega \in \widehat{\mathcal{D}}$ and let $L \in \mathcal{B}_0^\star$. We aim for constructing a function $g \in A^1_\omega$ such that $L(f) = L_g(f) = \langle f, g\rangle_{A^2_\omega}$ for all $f \in \mathcal{B}_0$, and $\|g\|_{A^1_\omega} \lesssim \|L\|$. To do this, let $g(z) = \sum_{n=0}^\infty \widehat{g}(n)z^n$, where $\widehat{g}(n) = \frac{\overline{L(z^n)}}{\omega_{2n+1}}$ for all $n \in \mathbb{N} \cup \{0\}$. Then

$$|\widehat{g}(n)| \leqslant \frac{\|L\|\|z^n\|_{\mathcal{B}}}{\omega_{2n+1}} \lesssim \frac{1}{\omega_{2n+1}}, \quad n \in \mathbb{N} \cup \{0\},$$

and therefore $g \in \mathcal{H}(\mathbb{D})$. If $f \in \mathcal{B}$ with $f(z) = \sum_{n=0}^\infty \widehat{f}(n)z^n$, then $f_r \in \mathcal{B}_0$ for each $r \in (0,1)$, and

$$\sum_{n=0}^m \widehat{f}(n)(rz)^n \to f_r(z), \quad m \to \infty,$$

in $(\mathcal{B}_0, \|\cdot\|_{\mathcal{B}})$. Therefore

$$L\left(\sum_{n=0}^m \widehat{f}(n)(rz)^n\right) = \sum_{n=0}^m \widehat{f}(n)\overline{\widehat{g}(n)}\omega_{2n+1}r^n \to L(f_r), \quad m \to \infty, \tag{4.5}$$

which implies

$$L(f_r) = \sum_{n=0}^\infty \widehat{f}(n)\overline{\widehat{g}(n)}\omega_{2n+1}r^n = \langle f, g_r\rangle_{A^2_\omega}, \tag{4.6}$$

and hence

$$\left|\langle f, g_r\rangle_{A^2_\omega}\right| = |L(f_r)| \leqslant \|L\|\|f_r\|_{\mathcal{B}} \leqslant \|L\|\|f\|_{\mathcal{B}}, \quad f \in \mathcal{B}. \tag{4.7}$$

Moreover, we just showed that $(A^1_\omega)^\star \simeq P_\omega(L^\infty(\mathbb{D}))$, and hence each $\Psi \in (A^1_\omega)^\star$ is of the form $\Psi(G) = \langle G, P_\omega(h)\rangle_{A^2_\omega}$ for some $h \in L^\infty(\mathbb{D})$ with $\|\Psi\| = \|h\|_{L^\infty(\mathbb{D})}$. Therefore, as $P_\omega : L^\infty(\mathbb{D}) \to \mathcal{B}$ is bounded by the first part of the proof, for each $\Psi \in (A^1_\omega)^\star$ there exists $f \in \mathcal{B}$ such that $\Psi(G) = \langle G, f\rangle_{A^2_\omega}$ with $\|f\|_{\mathcal{B}} \leqslant \|P_\omega\|_{L^\infty(\mathbb{D})\to\mathcal{B}}\|\Psi\|$, and hence a known byproduct of the Hahn-Banach theorem implies

$$\|g_r\|_{A^1_\omega} = \sup_{\Psi\in(A^1_\omega)^\star} \frac{|\Psi(g_r)|}{\|\Psi\|} \leqslant \|P_\omega\|_{L^\infty(\mathbb{D})\to\mathcal{B}} \sup_{f\in\mathcal{B}\setminus\{0\}} \frac{|\langle f, g_r\rangle_{A^2_\omega}|}{\|f\|_{\mathcal{B}}}. \tag{4.8}$$

By combining (4.7) and (4.8), and then letting $r \to 1^-$, we deduce $\|g\|_{A^1_\omega} \leqslant \|P_\omega\|_{L^\infty(\mathbb{D})\to\mathcal{B}}\|L\|$. Finally, by using that $\lim_{r\to 1^-} \|f_r - f\|_{\mathcal{B}} = 0$ for all $f \in \mathcal{B}_0$, and (4.6) we deduce

$$L(f) = \lim_{r\to 1} L(f_r) = \lim_{r\to 1} \sum_{n=0}^\infty \widehat{f}(n)\overline{\widehat{g}(n)}\omega_{2n+1}r^n = \langle f, g\rangle_{A^2_\omega}, \quad f \in \mathcal{B}_0.$$

Conversely, assume (iii). For each $n \in \mathbb{N} \cup \{0\}$, consider $L_n \in \mathcal{B}_0^\star$ defined by $L_n(f) = \widehat{f}(n)\omega_{2n+1}$, where $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$. There exists an absolute constant $C > 0$ such that

$$|L_n(f)| \leqslant C\|f\|_{\mathcal{B}}\omega_{2n+1}, \quad n \in \mathbb{N} \cup \{0\},$$

and hence $\|L_n\| = \sup_{f\in\mathcal{B}_0, f\neq 0} \frac{|L_n(f)|}{\|f\|_{\mathcal{B}}} \leqslant C\omega_{2n+1}$. Moreover, $2L_n(f) = \langle f, e_n\rangle_{A^2_\omega}$, where $e_n(z) = z^n$. Therefore the hypothesis (iii) yields

$$\omega_{n+1} = \|e_n\|_{A^1_\omega} \lesssim \|L\| \lesssim \omega_{2n+1}, \quad n \in \mathbb{N} \cup \{0\},$$

and thus $\omega \in \widehat{\mathcal{D}}$, by Lemma 6.

□

Acknowledgements: These notes have been written using previous tex-files of published papers, preprints and notes together with my collaborators Jouni Rättyä and Elena de la Rosa, so I thanks them for their permission to use this material.

References

[1] J. M. Anderson, J. Clunie and Ch. Pommerenke, On Bloch functions and normal functions, J. Reine Angew. Math. 270 (1974), 12–37.
[2] P. Duren, Theory of H^p Spaces, Academic Press, New York-London 1970.
[3] P. L. Duren and A. P. Schuster, Bergman Spaces, Math. Surveys and Monographs, Vol. 100, American Mathematical Society: Providence, Rhode Island, 2004.
[4] J. Garnett, Bounded analytic functions, Academic Press, New York, 1981.
[5] H. Hedenmalm, B. Korenblum and K. Zhu, Theory of Bergman Spaces, Graduate Texts in Mathematics, Vol. 199, Springer, New York, Berlin, etc. 2000.
[6] M. Jevtić and M. Pavlović, On multipliers from H^p to l^q, $0 < q < p < 1$, Arch. Math. (Basel) 56 (1991), no. 2, 174–180.
[7] M. Pavlović, Introduction to function spaces on the Disk, Posebna Izdanja [Special Editions], vol. **20**, Matematički Institut SANU, Beograd, 2004. *available online at:* http://f3.tiera.ru/2/M_Mathematics/MC_Calculus/ MCc_Complex%20variable/Pavlovic%20M.%20
[8] J. A. Peláez, Small weighted Bergman spaces, Proceedings of the summer school in complex and harmonic analysis, and related topics, (2016).
[9] J. A. Peláez and E. de la Rosa, Littlewood-Paley inequalities for fractional derivative on Bergman spaces, Ann. Fenn. Math. 47, (2022), 1109-1130.
[10] J. A. Peláez and J. Rättyä, Weighted Bergman spaces induced by rapidly increasing weights, Mem. Amer. Math. Soc. 227 (2014), no. 1066.
[11] J. A. Peláez and J. Rättyä, Two weight inequality for Bergman projection, J. Math. Pures Appl. 105 (2016), 102–130.
[12] J. A. Peláez and J. Rättyä, Trace class criteria for Toeplitz and composition operators on small Bergman spaces. Adv. Math. 293 (2016), 606–643.
[13] J.Á. Peláez and J. Rättyä, Bergman projection induced by radial weight, Adv. Math. 391 (2021). Paper No. 107950, 70 pp.
[14] J. Rättyä, Weighted Bergman spaces and doubling weights, unpublished notes.
[15] K. Zhu, Operator Theory in Function Spaces, Second Edition, Math. Surveys and Monographs, Vol. 138, American Mathematical Society: Providence, Rhode Island, 2007.

Departamento de Análisis Matemático, Universidad de Málaga, Campus de Teatinos, 29071 Málaga, Spain
Email address: `japelaez@uma.es`